

ISOMORPHIC BUSEMANN-PETTY PROBLEM FOR SECTIONS OF PROPORTIONAL DIMENSIONS

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ABSTRACT. The main result of this note is a solution to the isomorphic Busemann-Petty problem for sections of proportional dimensions, as follows. Suppose that $0 < \lambda < 1$, $k > \lambda n$, and K, L are origin-symmetric convex bodies in \mathbb{R}^n satisfying the inequalities

$$|K \cap H| \leq |L \cap H|, \quad \forall H \in Gr_{n-k},$$

where Gr_{n-k} is the Grassmanian of $(n-k)$ -dimensional subspaces of \mathbb{R}^n , and $|K|$ stands for volume of proper dimension. Then

$$|K|^{\frac{n-k}{n}} \leq C^k \left(\sqrt{\frac{(1 - \log \lambda)^3}{\lambda}} \right)^k |L|^{\frac{n-k}{n}},$$

where C is an absolute constant.

1. INTRODUCTION

The Busemann-Petty problem, raised in 1956 in [BP], asks the following question. Suppose that K, L are origin-symmetric convex bodies in \mathbb{R}^n so that the $(n-1)$ -dimensional volume of every central hyperplane section of K is smaller than the same for L , i.e.

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp|, \quad \forall \xi \in S^{n-1}. \quad (1)$$

Does it follow that the n -dimensional volume of K is smaller than that of L , i.e.

$$|K| \leq |L| ?$$

Here $\xi^\perp = \{x \in \mathbb{R}^n : (x, \xi) = 0\}$ is the central hyperplane perpendicular to ξ , and $|K|$ stands for volume of proper dimension. The problem was solved in the end of the 1990's as the result of a sequence of papers [LR], [Ba2], [Gi], [Bo4], [L], [Pa], [G1], [G2], [Z1], [Z2], [K1], [K2], [Z3], [GKS] ; see [K4, p. 3] or [G3, p. 343] for details. The answer is affirmative if $n \leq 4$, and it is negative if $n \geq 5$.

The lower dimensional Busemann-Petty problem asks the same question for sections of lower dimensions. Suppose K, L are origin-symmetric convex bodies in \mathbb{R}^n , and $1 \leq k \leq n-1$. Let Gr_{n-k} be the Grassmanian

of $(n - k)$ -dimensional subspaces of \mathbb{R}^n , and suppose that

$$|K \cap H| \leq |L \cap H|, \quad \forall H \in Gr_{n-k}. \quad (2)$$

Does it follow that $|K| \leq |L|$? It was proved in [BZ] (see also [K3], [K4, p.112], [RZ] and [M] for different proofs) that the answer is negative if the dimension of sections $n - k > 3$. The problem is still open for two- and three-dimensional sections ($n - k = 2, 3$, $n \geq 5$).

Since the answer to the Busemann-Petty problem is negative in most dimensions, it makes sense to ask the isomorphic Busemann-Petty problem, namely, does there exist an absolute constant C such that inequalities (1) imply

$$|K| \leq C |L| ?$$

If the answer to the isomorphic Busemann-Petty problem was affirmative, then by iteration there would exist an absolute constant C such that for every $1 \leq k \leq n - 1$

$$|K|^{\frac{n-k}{n}} \leq C^k |L|^{\frac{n-k}{n}}. \quad (3)$$

However, the isomorphic Busemann-Petty problem is still open and equivalent to the slicing problem [Bo1, Bo2, Ba1, MP], another major open problem in convex geometry. The slicing problem asks whether there exists an absolute constant C so that for any origin-symmetric convex body K in \mathbb{R}^n of volume 1 there is a hyperplane section of K whose $(n - 1)$ -dimensional volume is greater than $1/C$. In other words, does there exist an absolute constant C so that for any $n \in \mathbb{N}$ and any origin-symmetric convex body K in \mathbb{R}^n

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|. \quad (4)$$

The best current result $C \leq O(n^{1/4})$ is due to Klartag [Kl], who removed the logarithmic term from an earlier estimate of Bourgain [Bo3]. We refer the reader to [BGVV] for the history and partial results.

Iterating (4) one gets the lower dimensional slicing problem asking whether the inequality

$$|K|^{\frac{n-k}{n}} \leq C^k \max_{H \in Gr_{n-k}} |K \cap H| \quad (5)$$

holds with an absolute constant C , where $1 \leq k \leq n - 1$. Inequality (5) was recently proved in [K5] in the case where $k \geq \lambda n$, $0 < \lambda < 1$, with the constant $C = C(\lambda)$ dependent only on λ .

Proposition 1. ([K5, Corollary 3]) *There exists an absolute constant C such that for every $n \in \mathbb{N}$, every $0 < \lambda < 1$, every $k > \lambda n$, and every*

origin-symmetric convex body K in \mathbb{R}^n

$$|K|^{\frac{n-k}{n}} \leq C^k \left(\sqrt{\frac{(1 - \log \lambda)^3}{\lambda}} \right)^k \max_{H \in Gr_{n-k}} |K \cap H|.$$

In this note we prove an isomorphic Busemann-Petty problem for sections of proportional dimensions.

Theorem 1. *Suppose that $0 < \lambda < 1$, $k > \lambda n$, and K, L are origin-symmetric convex bodies in \mathbb{R}^n satisfying the inequalities*

$$|K \cap H| \leq |L \cap H|, \quad \forall H \in Gr_{n-k}.$$

Then

$$|K|^{\frac{n-k}{n}} \leq C^k \left(\sqrt{\frac{(1 - \log \lambda)^3}{\lambda}} \right)^k |L|^{\frac{n-k}{n}},$$

where C is an absolute constant.

It is easy to see that Theorem 1 implies Proposition 1; see Remark 2. It is not known whether Theorem 1 can be deduced from Proposition 1, we provide an independent proof here.

Proposition 1 was proved in [K5] for arbitrary measures in place of volume. The arguments of this paper do not allow for an extension of Theorem 1 to arbitrary measures, and, therefore, the possibility of such an extension remains open. Note that a version of the isomorphic Busemann-Petty problem for arbitrary measures was established in [KZ], but with the constant $C = \sqrt{n}$ depending on the dimension.

2. PROOF OF THEOREM 1

We need several definitions and facts. A closed bounded set K in \mathbb{R}^n is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K , and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on \mathbb{R}^n .

We use the polar formula for volume of a star body

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \quad (6)$$

The solution of the original Busemann-Petty problem was based on a connection with intersection bodies found by Lutwak [L]. In this paper we use a more general class of generalized intersection bodies

introduced by Zhang [Z4] in connection with the lower dimensional Busemann-Petty problem.

For $1 \leq k \leq n-1$, the $(n-k)$ -dimensional spherical Radon transform $R_{n-k} : C(S^{n-1}) \rightarrow C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1} \cap H} g(x) dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$. By the polar formula for volume, for every $H \in Gr_{n-k}$, we have

$$|K \cap H| = \frac{1}{n-k} \int_{S^{n-1} \cap H} \|x\|_K^{-n+k} dx = \frac{1}{n-k} R_{n-k}(\|\cdot\|_K^{-n+k})(H). \quad (7)$$

We say that an origin symmetric star body K in \mathbb{R}^n is a *generalized k -intersection body*, and write $K \in \mathcal{BP}_k^n$, if there exists a finite Borel non-negative measure μ on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_K^{-k} g(x) dx = \int_{Gr_{n-k}} R_{n-k}g(H) d\mu(H). \quad (8)$$

When $k=1$ we get the original class of intersection bodies introduced by Lutwak in [L].

For a star body K in \mathbb{R}^n and $1 \leq k < n$, denote by

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) = \inf \left\{ \left(\frac{|D|}{|K|} \right)^{1/n} : K \subset D, D \in \mathcal{BP}_k^n \right\}$$

the outer volume ratio distance from K to the class \mathcal{BP}_k^n . This quantity is directly related to the isomorphic Busemann-Petty problem.

Theorem 2. *Suppose that $1 \leq k \leq n-1$, and K, L are origin-symmetric star bodies in \mathbb{R}^n such that*

$$|K \cap H| \leq |L \cap H|, \quad \forall H \in Gr_{n-k}.$$

Then

$$|K|^{\frac{n-k}{n}} \leq (\text{o.v.r.}(K, \mathcal{BP}_k^n))^k |L|^{\frac{n-k}{n}}.$$

Proof : Let $s > \text{o.v.r.}(K, \mathcal{BP}_k^n)$, then there exists a star body $D \in \mathcal{BP}_k^n$ such that $K \subset D$ and

$$|D|^{\frac{1}{n}} \leq s |K|^{\frac{1}{n}}. \quad (9)$$

Let μ be the measure on Gr_{n-k} corresponding to D by definition (8).

By (7), the condition $|K \cap H| \leq |L \cap H|$ can be written as

$$R_{n-k}(\|\cdot\|_K^{-n+k})(H) \leq R_{n-k}(\|\cdot\|_L^{-n+k})(H), \quad \forall H \in Gr_{n-k}.$$

Integrating this inequality over Gr_{n-k} with respect to the measure μ and using (8) we get

$$\int_{S^{n-1}} \|x\|_D^{-k} \|x\|_K^{-n+k} dx \leq \int_{S^{n-1}} \|x\|_D^{-k} \|x\|_L^{-n+k} dx. \quad (10)$$

Since $K \subset D$, we have $\|x\|_K^{-k} \leq \|x\|_D^{-k}$, so the left-hand side of (10) can be estimated from below by

$$\int_{S^{n-1}} \|x\|_K^{-n} dx = n|K|.$$

By Hölder's inequality and (9), the right-hand side of (10) can be estimated from above by

$$\begin{aligned} \left(\int_{S^{n-1}} \|x\|_D^{-n} dx \right)^{\frac{k}{n}} \left(\int_{S^{n-1}} \|x\|_L^{-n} dx \right)^{\frac{n-k}{n}} &= n|D|^{\frac{k}{n}} |L|^{\frac{n-k}{n}} \\ &\leq ns^k |K|^{\frac{k}{n}} |L|^{\frac{n-k}{n}}. \end{aligned}$$

Combining these estimates and sending s to $\text{o.v.r.}(K, \mathcal{BP}_k^n)$, we get the result. \square

The outer volume ratio distance from a general convex body to the class of generalized k -intersection bodies was estimated in [KPZ].

Proposition 2. ([KPZ, Theorem 1.1]) *Let K be an origin-symmetric convex body in \mathbb{R}^n , and let $1 \leq k \leq n-1$. Then*

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) \leq C \sqrt{\frac{n}{k}} \left(\log \left(\frac{en}{k} \right) \right)^{3/2},$$

where C is an absolute constant.

Remark 1. In [KPZ, Theorem 1.1], the result was formulated with the logarithmic term raised to the power $1/2$ instead of $3/2$, due to a mistake. The correction was made in [K5].

Our main result immediately follows.

Proof of Theorem 1. Since $n/k < 1/\lambda$, by Proposition 2,

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) < C \sqrt{\frac{(1 - \log \lambda)^3}{\lambda}}.$$

The result follows from Theorem 2. \square

Remark 2. Theorem 1 implies Proposition 1 via a simple argument, similar to the one employed in [MP] for hyperplane sections. Indeed, assume that λ and k are as in Theorem 1, and suppose that

$$|K|^{\frac{n-k}{n}} = C(\lambda)^k |B_2^n|^{\frac{n-k}{n}}, \quad (11)$$

where

$$C(\lambda) = C \sqrt{\frac{(1 - \log \lambda)^3}{\lambda}},$$

B_2^n is the unit Euclidean ball in \mathbb{R}^n , and K is an origin-symmetric convex body in \mathbb{R}^n . Then, by Theorem 1, it is not possible that

$$|K \cap H| < |B_2^n \cap H| = |B_2^{n-k}|$$

for all $H \in Gr_{n-k}$, so

$$\max_{H \in Gr_{n-k}} |K \cap H| \geq |B_2^{n-k}|.$$

Dividing both sides by equal numbers from (11), we get

$$\frac{\max_{H \in Gr_{n-k}} |K \cap H|}{|K|^{\frac{n-k}{n}}} \geq \frac{|B_2^{n-k}|}{C(\lambda)^k |B_2^n|^{\frac{n-k}{n}}}.$$

By homogeneity, the condition (11) can be dropped, and the latter inequality holds for arbitrary origin-symmetric convex K . Now Proposition 1 follows from

$$\frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|} \in (e^{-k/2}, 1);$$

see for example [KL, Lemma 2.1].

Finally, we mention the following fact which can be proved by applying Theorem 1 twice.

Corollary 1. *Suppose that $0 < \lambda < 1$, $k > \lambda n$, $0 < c_1 < c_2$, and K, L are origin-symmetric convex bodies in \mathbb{R}^n such that*

$$c_1 \leq \frac{|L \cap H|}{|K \cap H|} \leq c_2, \quad \forall H \in Gr_{n-k}.$$

Then

$$\frac{c_1}{C^k \left(\sqrt{\frac{(1 - \log \lambda)^3}{\lambda}} \right)^k} \leq \frac{|L|^{\frac{n-k}{n}}}{|K|^{\frac{n-k}{n}}} \leq c_2 C^k \left(\sqrt{\frac{(1 - \log \lambda)^3}{\lambda}} \right)^k,$$

where C is an absolute constant.

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REFERENCES

- [Ba1] K. Ball, *Isometric problems in ℓ_p and sections of convex sets*, Ph.D. dissertation, Trinity College, Cambridge (1986).
- [Ba2] K. Ball, *Some remarks on the geometry of convex sets*, Geometric aspects of functional analysis (1986/87), Lecture Notes in Math. **1317**, Springer-Verlag, Berlin-Heidelberg-New York, 1988, 224–231.
- [Bo1] J. Bourgain, *On high-dimensional maximal functions associated to convex bodies*, Amer. J. Math. **108** (1986), 1467–1476.
- [Bo2] J. Bourgain, *Geometry of Banach spaces and harmonic analysis*, Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 871–878.
- [Bo3] J. Bourgain, *On the distribution of polynomials on high-dimensional convex sets*, Geometric aspects of functional analysis, Israel seminar (1989–90), Lecture Notes in Math. **1469** Springer, Berlin, 1991, 127–137.
- [Bo4] J. Bourgain, *On the Busemann-Petty problem for perturbations of the ball*, Geom. Funct. Anal. **1** (1991), 1–13.
- [BZ] J. Bourgain and Gaoyong Zhang, *On a generalization of the Busemann-Petty problem*. Convex Geometric Analysis, Math. Sci. Res. Inst. Publ. **34**, Cambridge Univ. Press, Cambridge, 1999, 53–58.
- [BGVV] S. Brazitikos, A. Giannopoulos, P. Valettas and B. Vritsiou, *Geometry of isotropic log-concave measures*, Amer. Math. Soc., Providence RI, 2014.
- [BP] H. Busemann and C. M. Petty, *Problems on convex bodies*, Math. Scand. **4** (1956), 88–94.
- [G1] R. J. Gardner, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994), 435–445.
- [G2] R. J. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, Annals of Math. **140** (1994), 435–447.
- [G3] R. J. Gardner, *Geometric tomography*, Second edition, Cambridge University Press, Cambridge, 2006.
- [GKS] R. J. Gardner, A. Koldobsky and Th. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Annals of Math. **149** (1999), 691–703.
- [Gi] A. Giannopoulos, *A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies*, Mathematika **37** (1990), 239–244.
- [Kl] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274–1290.
- [K1] A. Koldobsky, *Intersection bodies, positive definite distributions and the Busemann-Petty problem*, Amer. J. Math. **120** (1998), 827–840.
- [K2] A. Koldobsky, *Intersection bodies in \mathbb{R}^4* , Adv. Math. **136** (1998), 1–14.
- [K3] A. Koldobsky, *A functional analytic approach to intersection bodies*. Geom. Funct. Anal., **10** (2000), 1507–1526.
- [K4] A. Koldobsky, *Fourier analysis in convex geometry*, Amer. Math. Soc., Providence RI, 2005.

- [K5] A. Koldobsky, *Slicing inequalities for measures of convex bodies*, arXiv:1412.8550
- [KL] A. Koldobsky and M. Lifshits, *Average volume of sections of star bodies*, Geometric Aspects of Functional Analysis, V. Milman and G. Schechtman, eds., Lecture Notes in Math. **1745** (2000), 119–146.
- [KPZ] A. Koldobsky, G. Paouris and M. Zymonopoulou, *Isomorphic properties of intersection bodies*, J. Funct. Anal. **261** (2011), 2697–2716.
- [KZ] A. Koldobsky and A. Zvavitch, *An isomorphic version of the Busemann-Petty problem for arbitrary measures*, Geom. Dedicata **174** (2015), 261–277
- [LR] D. G. Larman and C. A. Rogers, *The existence of a centrally symmetric convex body with central sections that are unexpectedly small*, Mathematika **22** (1975), 164–175.
- [L] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.
- [M] E. Milman, *Generalized intersection bodies*. J. Funct. Anal., **240** (2) (2006), 530–567.
- [MP] V. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n -dimensional space*, in: Geometric Aspects of Functional Analysis, ed. by J. Lindenstrauss and V. Milman, Lecture Notes in Mathematics **1376**, Springer, Heidelberg, 1989, pp. 64–104.
- [Pa] M. Papadimitrakis, *On the Busemann-Petty problem about convex, centrally symmetric bodies in \mathbb{R}^n* , Mathematika **39** (1992), 258–266.
- [RZ] B. Rubin and Gaoyong Zhang, *Generalizations of the Busemann-Petty problem for sections of convex bodies*, J. Funct. Anal. **213** (2004), 473–501.
- [Z1] Gaoyong Zhang, *Centered bodies and dual mixed volumes*, Trans. Amer. Math. Soc. **345** (1994), 777–801.
- [Z2] Gaoyong Zhang, *Intersection bodies and Busemann-Petty inequalities in \mathbb{R}^4* , Annals of Math. **140** (1994), 331–346.
- [Z3] Gaoyong Zhang, *A positive answer to the Busemann-Petty problem in four dimensions*, Annals of Math. **149** (1999), 535–543.
- [Z4] Gaoyong Zhang, *Sections of convex bodies*, Amer. J. Math. **118** (1996), 319–340.

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